ROTATION OF A THIN CYLINDER IN A VISCOPLASTIC MEDIUM WITH HEAT TRANSFER

E. A. Leonova

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Self-similar solutions of the axisymmetric problem of rotation of a linear viscoplastic medium were found in [1, 2]. This paper presents the solution of the self-similar problem of the flow with heat transfer of a heat-conducting viscoplastic medium for which the shear stress is a function of the shear strain rate and temperature.

An infinitely long cylinder of small radius r_0 rotates in an infinite medium initially at rest with constant initial temperatue T_0. There is a heat flow through the surface of the cylinder such that the quantity of heat withdrawn by the cylinder in unit time per unit length is constant and equal to Q_0. At infinity the medium remains at rest and has the temperature T_0. The flow and propagation of heat in the medium are described without allowance for dissipation by the system of equations

$$\rho\frac{\partial v}{\partial t}-\frac{\partial\Phi}{\partial\varepsilon}\left(\frac{\partial^2 v}{\partial r^2}-\frac{1}{r}\frac{\partial v}{\partial r}+\frac{v}{r^2}\right)+\frac{\partial\Phi}{\partial T}\frac{\partial T}{\partial r}+2\frac{\Phi}{r}=0,$$

$$\frac{\partial T}{\partial t}-a^2\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)=0,\qquad \Phi=\Phi\left(\frac{\varepsilon}{\varepsilon_0},\ \frac{T}{T_0}\right),$$

$$\varepsilon=-\frac{\partial v}{\partial r}+\frac{v}{r}\qquad (\varepsilon_0,\ T_0=\text{const}),\tag{1}$$

where v is the velocity, T is temperature, t is time, ρ is density, a^2 is the thermal diffusivity of the medium, and Φ is a function relating the shear stress to the shear strain rate and temperature.

The initial and boundary conditions are

$$T(r,0)=T_0,\qquad \lim_{r_0\to 0}\left(-2\pi r_0\lambda\frac{\partial T}{\partial r}\right)=Q_0,$$

$$\lim_{r\to\infty}T(r,t)=T_0,\tag{2}$$

$$v(r,0)=0,\qquad \lim_{r\to\infty}v(r,t)=0,\qquad \lim_{r\to\infty}\frac{\partial v}{\partial r}=0,\tag{3}$$

$$\lim_{r_0\to 0}\left[\frac{d}{dt}\left(\frac{\pi\rho}{2}r_0^4\omega\right)+2\pi r_0^2\Phi\right]=M_0 t.\tag{4}$$

Condition (4) is the law of motion of the cylinder, ω is the angular velocity, and M_0t is the applied moment. We introduce $\tau_0(\theta)$ and $\tau(\theta,\gamma)$ so that

$$\Phi=\tau_s[\tau_0(\theta)+\tau(\theta,\ \gamma)],\quad \tau(0,\ \theta)=0,$$

$$\theta=T/T_0,\qquad \gamma=\varepsilon/\varepsilon_0.\tag{5}$$

It can be shown that the solution of problem (1)–(4) has the form

$$T=T_0\,\theta(\xi),\quad v=a\varepsilon_0 t^{1/2}u(\xi),\quad \xi=r/2a\sqrt{t}.$$

Here, the dimensionless $\theta(\xi)$ and $u(\xi)$ are found from the system of equations

$$\frac{\partial\tau}{\partial\gamma}\left(\frac{d^2u}{d\xi^2}-\frac{1}{\xi}\frac{du}{d\xi}+\frac{u}{\xi^2}\right)-2\frac{\partial\tau}{\partial\theta}\frac{d\theta}{d\xi}-4\frac{\tau}{\xi}+$$

$$+p\left(\xi\frac{du}{d\xi}-u\right)-2\frac{d\tau_0}{d\theta}\frac{d\theta}{d\xi}-4\frac{\tau_0}{\xi}=0,\tag{6}$$

$$\gamma=\frac{1}{2}\left(-\frac{du}{d\xi}+\frac{u}{\xi}\right),\qquad p=\frac{2\rho\varepsilon_0 a^2}{\tau_s},$$

$$\frac{d^2\theta}{d\xi^2}+\left(\frac{1}{\xi}+2\xi\right)\frac{d\theta}{d\xi}=0.\tag{7}$$

Integration of Eq. (7) with boundary conditions corresponding to (2)

$$\lim_{\xi\to\infty}\theta=1,\qquad \lim_{\xi\to 0}(\xi d\theta/d\xi)=-2Q,\qquad Q=Q_0/4\pi\lambda T_0,$$

gives the temperature field

$$\theta(\xi)=1-Q\,\mathrm{Ei}(-\xi^2),$$

$$\mathrm{Ei}(-z)=-\int_z^\infty\frac{e^{-z}}{z}dz<0\quad\text{for }0<z<\infty.\tag{8}$$

Now the problem reduces to finding the solution of system (6). We introduce the function F from (5): $\gamma=F(\tau,\theta)$. This makes it possible to reduce (6) to the form

$$\frac{d\tau}{d\xi}+2\frac{\tau}{\xi}+p\xi F(\tau,\ \theta)+\frac{d\tau_0}{d\theta}\frac{d\theta}{d\xi}+2\frac{\tau_0}{\xi}=0,\tag{9}$$

$$du/d\xi-u/\xi+2F(\tau,\ \theta)=0.\tag{10}$$

We will investigate Eq. (9) for cooling of the medium by the cylinder ($Q<0$); the corresponding case of heating is easily obtained. Through each point on the plane $\xi\tau$ in the region $\xi\ge 0$, $\tau\ge 0$, determined by the conditions of the problem, there passes one and only one integral curve of Eq. (9), except for two singular points located in an infinitely remote part of the plane on the coordinate axes. In the case of a nonzero yield stress $\tau_0(\theta)$ all the curves intersect the ξ axis at finite points at a certain angle, none of the integral curves leaves the finite part of the plane in the direction of the singular point $(\infty, 0)$. At $\tau_0(\theta)\equiv 0$ all the curves tend to that point; the point $(0, 0)$ is a singular point of the saddle type, the axes $\xi=0$, $\tau=0$ are integral curves. It is important to study the behavior of the solution for the common experimental case $\tau_0(\theta)=B\exp(-\varkappa_0\theta)$ and $F(\tau,\theta)$ increasing at large τ as $F(\tau,\theta)=A\exp(\varkappa_1\theta)\,\tau^N$ ($\varkappa_0, \varkappa_1>0$, $N>1$); henceforth we will assume that $B\ne 0$. The behavior of the integral curves near the τ axis depends importantly on the parameters of the equation. According to the values of the parameters we can distinguish the following two essentially different cases of distribution of the integral curves.

1. The case $1+\varkappa_0 Q\ge 0$.

The integral curves can all be divided into two classes. The curves belonging to the first class have vertical asymptotes. If $N<2-\varkappa_1 Q$ the second class consists of a bundle of monotonically decreasing curves, which as $\xi\to 0$ behave as follows:

$$\tau=E\xi^{-2}+o(\xi^{-2}),\qquad E>0.\tag{11}$$

The curve Ω separating these classes goes to infinity as $\xi\to 0$, so that

$$\tau=D_1\xi^{-\nu_1}+o(\xi^{-\nu_1}),\qquad \nu_1=2\frac{1-\varkappa_1 Q}{N-1},$$

$$D_1=\left[2\frac{2-\varkappa_1 Q-N}{(N-1)Ap}\right]^{1/(N-1)}\exp\left(\varkappa_1\frac{Qb-1}{N-1}\right),$$

$$(b=0.5772\ldots).\tag{12}$$

However, if $N\ge 2-\varkappa_1 Q$, there is no second class and all the curves belong to the first class.

2. The case $1+\varkappa_0 Q<0$.

Again there are two classes. The integral curves of the first class behave in the same way as the corresponding curves in the first case. The curves of the second class begin at finite points on the ξ axis, increase with increase in ξ, pass through a maximum, and then decrease. For the separating curve Ω, which goes to infinity as $\xi \to 0$, we have either relation (12) when $-N\varkappa_0 Q < 1 - \varkappa_0 Q - \varkappa_1 Q$ or

$$\tau = D_2 \xi^{-\nu_2} + o(\xi^{-\nu_2}), \qquad \nu_2 = 2\frac{1 - \varkappa_0 Q - \varkappa_1 Q}{N},$$

$$D_2 = \left[-\frac{(1+\varkappa_0 Q)B}{pA}\right]^{1/N} \exp\left[\frac{(\varkappa_0+\varkappa_1)(Qb-1)}{N}\right], \tag{13}$$

when $-N\varkappa_0 Q > 1 - \varkappa_1 Q - \varkappa_0 Q$.

The field of integral curves is shown in Fig. 1.

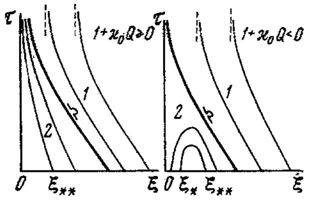

Fig. 1

We will now consider the possibilities of satisfying boundary conditions (3) and (4). Conditions (3) require that

$$\lim \tau = 0, \quad \lim u = 0 \quad \text{as} \quad \xi \to 0\,. \tag{14}$$

It is easy to see that the first condition is satisfied for all the integral curves. Each of them can be made to correspond with $u(\xi)$ by integrating (10). In this case it is possible to satisfy the second of conditions (14)

$$u = 2\xi \int\limits_{\xi}^{\xi_{**}} \frac{F(\tau, \theta)}{\xi}\, d\xi, \tag{15}$$

where ξ_{**} is the point of intersection of the integral curve and the axis of abscissas (Fig. 1).

Turning to (4), we see that it is not satisfied by functions represented by integral curves of the first class. In fact, each such curve is located in the region $\xi > \xi^\circ$ (ξ° is its asymptote). However, the assumption that part of the medium adjacent to the cylinder moves with it as a solid leads to the necessity of satisfying the condition

$$\lim_{\xi\to\xi^\circ}\left[\frac{1}{2}p\xi^3 u(\xi) + \xi^2\tau_0(\theta) + \xi^2\tau\right] = M, \qquad M = \frac{M_0}{8\pi a^2 \tau_s}, \tag{16}$$

which is violated for curves of the first class since $\tau \to \infty$ as $\xi \to \xi^\circ$, and all the terms are positive.

For the Ω curves from (12) and (13) we have

$$\lim \xi^2(\tau_0(\theta) + \tau) = +\infty \quad \text{as} \quad \xi \to 0\,.$$

Hence it follows that condition (4), which reduces to the form

$$\lim\,[(1/4)\,p\xi^4\,F(\tau, \theta) + \xi^2\tau_0(\theta) + \xi^2\tau] = M \quad \text{as} \quad \xi \to 0 \tag{17}$$

is not satisfied. For curves of the second class in case (1) we have

$$\lim_{\xi\to 0}\xi^4 F(\tau, \theta) = 0, \qquad \lim_{\xi\to 0}\xi^2\tau(\xi) = c, \qquad 0 < c < \infty,$$

$$\lim_{\xi\to 0}\tau_0(\theta)\,\xi^2 = \begin{cases} B\exp\,[-\varkappa_0(1+Qb)] & \text{at} \quad 1+\varkappa_0 Q = 0, \\ 0 & \text{at} \quad 1+\varkappa_0 Q > 0\,. \end{cases}$$

Consequently, to each value in the range $0 < M < \infty$ there corresponds one and only one curve. Each curve of the second class gives a unique solution of the problem.

In case (2) for curves of the second class condition (4) can be satisfied only in the form (16). For the integral curve starting from some point ξ_* on the ξ axis, for which $(d\tau/d\xi)_{\xi_*} > 0$, from (16) and (15) we have

$$p\int\limits_{\xi_*}^{\xi_{**}(\xi_*)} \frac{F(\tau, \theta)}{\xi}\, d\xi = \frac{M - \tau_0(\xi_*)\,\xi_*^2}{\xi_*^4}\,. \tag{18}$$

It can be shown that if $M < \tau_0(\xi_0)\xi_0^2$, $\xi_0^2 = \ln(-\varkappa_0 Q)$, the equation has no solution, since $\tau_0(\xi_*)\xi_*^2 > \tau_0(\xi_0)\xi_0^2$ for any ξ_*. However, if

$$M > \tau_0(\xi_0)\,\xi_0^2\,, \tag{19}$$

then for any M satisfying this condition there is a unique root ξ_* of Eq. (18), and $\xi_{00} < \xi_* < \xi_0$, where ξ_{00} is the solution of the equation $\tau_0(\xi)\xi^2 = M$.

In fact, this is proved by the graphs of the functions

$$J(\xi) = \int\limits_{\xi}^{\xi_{**}(\xi)} \frac{F(\tau, \theta)}{\xi}\, d\xi, \qquad J_1(\xi) = \frac{M - \xi^2\tau_0(\xi)}{\xi^4},$$

presented in Fig. 2. Thus, in this case to each value of M, bounded by condition (19), where corresponds one and only one integral curve of Eq. (9) belonging to the second class and giving a unique solution of the problem. Once the stress distribution has been found, the velocity field is determined from expression (15).

Thus, depending on the properties of the medium and the values of the external moment two different types of motion are possible. In case (1), which for the given medium corresponds to a relatively small cooling rate, flow develops directly at the cylinder at any value of the external moment, the flow zone increases with time, constantly bordering the cylinder, or no region of the medium, other than the cylinder itself, is involved in the motion. In case (2), corresponding to intense heat transfer, flow develops if the external moment exceeds a certain value. A rigid layer, increasing with time according to the law $r_*(t) = 2a\xi_*\sqrt{t}$, is formed on the cylinder. In this case the growing cylinder rotates at a constant angular velocity $\omega = \varepsilon_0 u(\xi_*)/2\xi_*$.

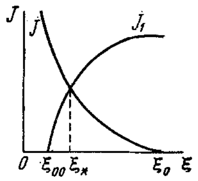

Fig. 2

The solution obtained makes it possible to determine the yield point parameters of the medium from the values of the quantity of heat withdrawn in unit time and the applied moment. Noting the value of $Q = Q_*$ at which a rigid layer first appears on the cylinder, we obtain the parameter

$$\varkappa_0 = 1/Q_*\,.$$

Then, measuring the minimum moment M_{min} capable of causing flow under the conditions $1 + \varkappa_0 Q < 0$, we find

$$B = \frac{M_{min}}{\ln(-\varkappa_0 Q)}\exp\,[-\varkappa_0 Q\,\mathrm{Ei}\,(-\ln(-\varkappa_0 Q)) + \varkappa_0]\,.$$

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